ADVANCED PARTIAL DIFFERENTIAL EQUATIONS: HOMEWORK 1

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1. Chapter 1, Problem 1

1. The Eikonal equation is first order and fully nonlinear.

2. The Nonlinear Poisson equation is second order semilinear.

3. The p-Laplacian is second order quasilinear. To see this, use the product rule:

$$\nabla \cdot (|Du|^{p-2} Du) = \nabla(|Du|^{p-2}) \cdot Du + |Du|^{p-2} \Delta u$$

4. The Minimal Surface equation is second order quasilinear. Again, use the product rule:

$$\nabla \cdot \Big(\frac{Du}{(1 + |Du|^2)^{1/2}} \Big) = \nabla \Big(\frac{1}{(1 + |Du|^2)^{1/2}} \Big) \cdot Du + \Big(\frac{1}{(1 + |Du|^2)^{1/2}} \Big) \Delta u$$

5. The Monge Ampere equation is second order fully nonlinear.

6. The Hamilton Jacobi Equation is first order fully nonlinear.

7. The Scalar Conservation law will in general be first order and fully nonlinear, depending on the properties of $\mathbf{F}(u)$.

8. The Inviscid Burger's equation is first order quasilinear.

9. The Scalar reaction-diffusion equation is second order semilinear.

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10. The Porous Medium equation is second order quasilinear.

11. The nonlinear wave equation is second order semilinear.

12. The KdV equation is third order semilinear.

13. The Nonlinear Schrodinger equation is second order semilinear.

2. Chapter 1, Problem 5

Set $g(t) := f(tx)$. Then, $g'(t) = x \cdot \nabla f(tx)$ by the chain rule. By induction we obviously have that $g^{(n)}(t) = (x \cdot \nabla)^n f(tx)$, where $x \cdot \nabla := x_1 \frac{\partial}{\partial x_i} + \dots + x_n \frac{\partial}{\partial x_n}$. Using Taylor's formula with remainder for a single variable, we see:

$$\begin{aligned} g(t) &= \sum_{i=0}^{k} \frac{g^{(i)}(0)t^i}{i!} + \mathcal{O}(t^{k+1}|x|^{k+1}) \\ &= \sum_{i=0}^{k} \frac{(x \cdot \nabla)^i f(0)t^i}{i!} + \mathcal{O}(t^{k+1}|x|^{k+1}) \end{aligned} \tag{2.1}$$

Now, note that $g(1) = f(x)$, so that

$$f(x) = \sum_{i=0}^{k} \frac{(x \cdot \nabla)^i f(0)}{i!} + \mathcal{O}(|x|^{k+1})$$

Using the result of problem 3 of this chapter, we employ the multinomial theorem with $y_i = x_i \frac{\partial}{\partial x_i}$ to find:

$$(x \cdot \nabla)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha D^\alpha$$

And hence plugging this into our sum:

$$
\begin{aligned}
f(x) &= \sum_{i=0}^{k} \sum_{|\alpha|=i} \frac{x^\alpha D^\alpha f(0)}{\alpha!} + \mathcal{O}(|x|^{k+1}) \\
&= \sum_{|\alpha|\leq k} \frac{x^\alpha D^\alpha f(0)}{\alpha!} + \mathcal{O}(|x|^{k+1})
\end{aligned} \tag{2.2}
$$

As desired.

3. Chapter 2, Problem 1

Multiply our equation by e^{ct} to find:

$$
\begin{aligned}
e^{ct}u_t + e^{ct}b \cdot Du + ce^{ct}u &= (e^{ct}u)_t + b \cdot D(e^{ct}u) \\
&= 0
\end{aligned} \tag{3.1}
$$

Set $e^{ct}u := v$. We see that $v(x,0) = g(x)$, and so following the method of solution presented in 2.1, we have:

$$v(x,t) = g(x - tb) \implies u(x,t) = e^{-ct}g(x - tb)$$

And for the nonhomogeneous case we can quickly find a solution:

$$v(x,t) = g(x - tb) + \int_0^t e^{cs} f(x + (s-t)b, s)ds$$

So that

$$u(x,t) = e^{-ct}g(x - tb) + \int_0^t e^{c(s-t)} f(x + (s-t)b, s)ds$$

Which yields the result.